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The Grammar of Science: Let's 'Log' (Part 1)

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No. I am not asking you to go "Rock'n Roll' with me at the "Let's Rock: The Retro Festival" in UK. And I am not asking you to go cut down trees in the forest. But I would like you to reflect back to the "logarithm" function in mathematics and statistics that you learned in your high school days. Many people get blocked out and start to not like statistics or mathematics when they see numbers get converted into "logarithm". Why do we have to do it in "log-scale"? Why not just use or analyze data in simple numbering scale? There are reasons behind this. And it is as fun as going rock and roll at the concert!

Back to basic - "Logarithm" is rockin!

You may think mathematics is fixed and full of unchanging rules and truths. But it may not be so. As an example of the "logarithm" you will be surprised that modern mathematicians will give very different definitions from the mathematicians several centuries ago[1]. If we go back through the history of "logarithm", we will see that the concept and methods had been evolved over time. Historically, the concept of logarithm was independently invented in the 17th century by at least two mathematicians, the Scottish John Napier (1550-1617) and the Swiss Joost Bürgi (1552-1632). In summary, logarithm was developed to speed up calculations, mainly to reduce the time required for multiplying numbers with many digits[2]. Napier was the first mathematician who named the term "logarithm" from the Greek roots - "logos" meaning proportion + "arithmus" meaning number; this is because he used it to relate numbers to another value when he wanted to calculate complex formula for multiplication of very large numbers in astronomy. Almost at the same time period, Burgi came up with the concept of logarithm similar to Napier when he tried to make mathematical operations simpler by combining multiplication, division, square roots, and cube roots together in one table. Burgi did not get much credit as a founder of the logarithm because he did not share his work at that time while Napier published his findings[3].

By mathematical definition, "logarithm" is the exponent or power to which a base must be raised to yield a given number[2]. That is, x is the logarithm of n to the base b if $b^x = n$; in which we can write: $x = \log_b n$. To make it clear, look at these two examples:

$2^3 = 8$, so we can say that 3 is the logarithm of 8 to base 2; or $3 = \log_2 8$

$10^2 = 100$, so we can say that 2 is the logarithm of 100 to base 10; or $2 = \log_{10} 100$

Perhaps many of us are more familiar with the second example, "logarithm base 10" which is called "common" or "Briggsian" logarithm. Historically, Napier's ideas were taken up and revised by the English mathematician Henry Briggs (1561-1630) who had invented the common logarithm table and made it accepted throughout Europe. Again, his innovation was used to solve the burden of mathematicians, astronomers, and other scientists in performing the long and tedious calculations[2]. The common table of logarithm base 10 concept is shown in figure 1.

Number	Logarithm	Antilogarithm
1	$\log_{10}(1) = 0$	$10^0 = 1$
10	$\log_{10}(10) = 1$	$10^1 = 10$
100	$\log_{10}(100) = 2$	$10^2 = 100$
1000	$\log_{10}(1000) = 3$	$10^3 = 1000$
10000	$\log_{10}(10000) = 4$	$10^4 = 10000$
100000	$\log_{10}(100000) = 5$	$10^5 = 100000$
1000000	$\log_{10}(10000000) = 6$	$10^6 = 1000000$
10000000	$\log_{10}(100000000) = 7$	$10^7 = 10000000$

Figure 1. Common Logarithm, Log base 10

The basic idea behind "logarithm" is that "addition and subtraction" are easier to perform than "multiplication and division" which Napier had said that the latter operation require a "tedious expenditure of time" and are subject to "slippery errors"[2]. As an example of the law of exponent, the multiplication of numbers could be presented as the exponents additively: i.e., $b^x b^y = b^{x+y}$. Thus by correlating the geometric sequence of numbers b, b^2, b^3,… (b = base) and the arithmetic sequence 1, 2, 3,…, we don't have to do series of "multiplication and division" but simply do "addition and subtraction". For the sake of simple example, we will express in terms using common logarithm base 10 (we can do similar operations with logarithms of other bases); the calculation of long and tedious numbers like 100000 x 1000000000 could be done by:

- Exponent or power base 10: $b^x b^y = b^{x+y}$

  100000 x 1000000000 = $10^5 \times 10^9 = 10^{5+9} = 10^{14}$ = 100000000000000

- Logarithm base 10: $\log_b mn = \log_b m + \log_b n$

  $\log_{10}(100000 \times 1000000000) = \log_{10}(100000) + \log_{10}(1000000000) = 5+9 = 14$,

  then convert back (so-called antilogarithm) = 10^{14} = 100000000000000

What is so natural about the "natural log"?

Besides "common" logarithm or "log base 10", when you look at different statistical procedures, you will see a lot of "natural" logarithm or "log base e". Let's go back a bit in time. Natural log was developed by Leonhard Euler (1707-1783) (pronounced "oiler") who was a mathematician of 18th century and is considered one of the greatest mathematicians of all time[2]. In fact, Euler studied with Johann Bernoulli ("Bernoulli" is another statistical term that you see a lot in statistics textbook – we may have another article on this later). Despite his blindness later in life, Euler had written nearly 900 books or produced on average one mathematical paper every week, covering almost all aspects of mathematics, from geometry to calculus to trigonometry to algebra to number theory, as well as optics, astronomy, cartography, mechanics, weights and measures and even the theory of music[4]. A lot of mathematical notation created or popularized by Euler included, for examples, e the base of the natural logarithm, f(x) the function f as applied to the variable or argument x, Σ sigma, the sum of total of a set of numbers, etc.

So what is "natural logarithm"?

The "natural log" is usually written as $\log_e x$ or ln(x). Why do they use "ln" not "nl" for "natural log"? One of the explanations is given that it comes from the Latin name is "logarithmusnaturali". And the natural log is the inverse of "e". The "e" is sometimes called "Euler's number" which Euler said that "e" is not his name but rather means "exponential". As one of the easy way to define "natural log", we can say that the natural log gives you the time needed to reach a certain level of growth[5].

ln(x) = amount of time needed to reach a certain level of continuous growth
= time needed to grow to x (with 100% continuous compounding)

e^x = amount of continuous growth after a certain amount of time
= amount of growth after time x (with 100% continuous compounding)

Again what is "e"? ("ln" is the inverse of "e")

One way to explain the "e" or the amount of growth is to think about the scenario of calculating interest growth of your deposited money (Figure 2). Suppose you open a bank account with $1 deposited and the interest rate of 100% per year growing continuously.

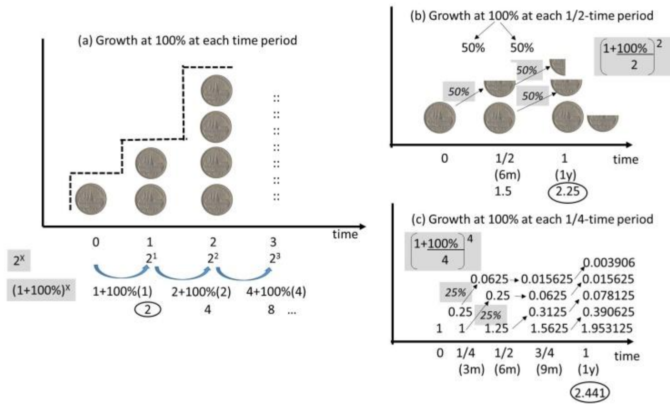

Figure 2. Growth and time of interest

As shown in figure 2(a), at the end of year 1, you will have $2; at year 2, you will have $4; at year 3, the money will increase to $8, and so on. The formula for increase could be written as $[1+100\%]^x$ where x = 1 time (1 year) to reach 100%.

But if the interest rate of 100% is given at two times-period over the year (thus 50% increase every half year), the money that you will get at the end of each year will be different. As shown in figure 2(b), you started at $1, but at the end of year 1, you will have $2.25, not $2. The formula for this increase is $[1+100\%/x]^x$ where x = 2 times (in 1 year) to reach 100%.

Again, if the interest rate of 100% is given at four times-period over the year (thus 25% increase quarterly), you money that you will get at the end of each year will also be different. As shown in figure 2(c), you started at $1, but at the end of year 1, you will have $2.441, not $2 or $2.25. The formula for this increase is $[1+100\%/x]^x$ where x = 4 times (in 1 year) to reach 100%.

If this is the case - how much growth will you get after many x units of time per year to reach 100% continuous growth? It seems that – “the larger number of x, the larger number of gain.” (When x=1, 2, 4, …, from initial \$1 with 100% growth per year, you get \$2, \$2.25, \$2.441.. at the end of the year, respectively!). If so, you should ask for x = 100 times per year to reach 100% growth rate so that at the end of the year you will get much higher than \$2. The bank will not accept that for sure! But wait – the bank has no such worry. It has been proved that when x is getting higher to a certain limit, the gain will be somewhat stable as shown in figure 3.

Time (x)	$\left(1+\frac{100\%}{x}\right)^x$
1	2
2	2.25
4	2.44140625
100	2.704813829
1000	2.716923932
10000	2.718145927
100000	2.718268237
1000000	2.718280469
10000000	2.718281694

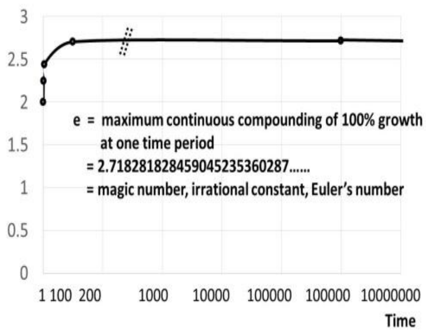

Figure 3. Natural logarithm, Log base e

As you can see, when x keeps getting larger and larger, the growth is slowing down. Or mathematically said - as the number of compounding increases, the computed value appears to be approaching some fixed value - "2.718281828459045235360287……". This number we're approaching is called "e". So “e” is defined as “the maximum continuous compounding of 100% growth at one time period”[6]. It is also called magic number, Euler’s number, or irrational constant. Irrational number means its value can't be given precisely in decimal notation. Like π in geometry, the ratio of the circumference of a circle to its diameter, its value is 22/7 or "3.141592653589……" which is another irrational number. So in any mathematical formula, we usually write this number as a letter-name (e or π) because that was easier[2].

The number "e" is the "natural" exponential, because it arises naturally in math and physical sciences. We may think the numbering system of base 10 (0..10..20..) that we are familiar is “natural” to us because almost all of us have 10 fingers. But we should say that the numbering base 10 is “common” to us; that is why logarithm base 10 is sometimes called “common log” as mentioned before. But in mathematical and scientific sense, there are several other bases that we also actually use but hardly realize it, for examples, base-60 in hours-minutes-seconds, base-12 in feet-inches, or based-2 (0-1 or on-off) or base-16 in computer science. Base 10 is good for counting in simple way, but it becomes more complicate when we monitor continuous growth like calculating interest rate in the banking example. The exponential functions are thus useful for modeling many systems that occur in our “natural” world[6,7]. It represents continuous growth in "real life" situations[6]. We will see that “e” is the “natural” base rate of growth of any systems or processes that grow continually and exponentially; for example, population, radioactive decay, interest, bacteria, and more. Even jagged systems that don’t grow smoothly can be approximated by “e”[8].

In brief, we can simply say that “e” and “ln” can tell us the relationship of growth and time[5] such that:

e^x where x is “time”, we will get growth at that time

ln(x) where x is “growth”, we will get time it would take to get that growth

For example:

e^4 = 54.59815

= After 4 units of time with 100% growth rate, we get the amount of growth increasing to 54.59815 times of the original amount that we start with.

(i.e., if we start with 1, it will increase to 54.59815 at the end of 4 units of time)

ln(54.59815) or l_e(54.59815) = 4

= If we want growth of 54.59615 times from what we started with, at the growth rate of 100%, we have to wait for 4 units of time.

However, the growth rate does not have to always be continually at 100%. It could be at any rate. The generic formula for exponential growth ise^{rt}; as an example of growth rate at 150% and timing is 4 units of time, we will get:

e^{rt} *where*	If growth rate = 150% and time = 4 unit
r = rate	then $e^{1.5x4}$ = e^6 = 403.42879
t = time period	or $\log_e$(403.42879) = Ln(403.42879) = 6

How “taking log” helps solve complex calculation

Based on its development, the logarithm has become a magic tool for mathematicians, physicists, and engineers used for simplifying complex calculations as it would make the multiplication and division of large numbers into an easier form of looking up values in a table and then adding them for addition and subtracting them for division[3]. This notation can generally apply to different kinds of log – “common log”, “natural log” or log of other bases. The following explanation will use “natural log” or “ln” as examples.

Let's start with basic properties of "log".

- ln(x) where x is "growth", we will get time it would take to get that growth[5].

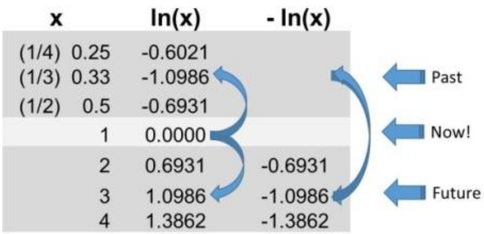

Figure 4. Basic properties of logarithm

What is ln(1)?

- ln(1) = 0
- As we want to have growth of 1 but 1 is our starting point (we put in $1 the bank and wait to get $1 in the bank!) – we don't have to wait, so time = 0.

What is ln(3)?

- ln(3) = 1.0986
- As we want to have growth of 3 when 1 is our starting point (we put in $1 the bank and wait to get $3 in the bank) – we have to wait 1.0986 units of time.

What is ln(1/3) or ln(0.33)?

- ln(0.33) = -1.0986
- Now we want to look at a fraction growth of 1/3 when 1 is our current point or reference and our continuously growth is still 100%. As ln(3) means we will get the amount three times from the current amount. So ln(1/3) means we have to inverse it; and it is equal to -1.0986. That means - if we have time machine going backwards to the past1.0986 units of time we would have 1/3 ($0.33) of our current amount of 1 ($1) today.
- As shown in figure 4, the value we get from ln(1/3) is equivalent to –ln(3).Thus, ln(1/x) = – ln(x)

What is ln(-x)?

- ln(-3) = impossible!
- It is impossible that money or others (say, bacteria) will grow from 1 to -3 or any other negative amount. (Note that in real life we may have negative money printed in red in bank account because we overspent from what we have and the bank allows us to do so before claiming that we are bankrupt! But the truth is we cannot have "negative" amount of money! Thus, ln(negative number) = undefined. "Undefined" means that there is no amount of time we can wait to get a negative amount.
- Using the same logic, the real logarithmic function ln(x) is defined only for x>0. We can't find a number x that would get $e^x = 0$. As that x does not exist, then ln(0) is also undefined.

Now, how logarithm turns "multiplication into addition" and "division into subtraction" (Figure 5).

- ln(x) – where x is "growth", we will get time it would take to get that growth[5]

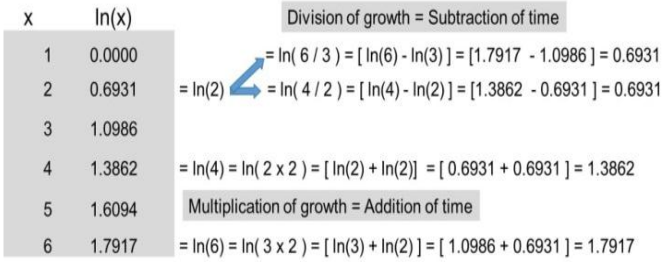

Figure 5. Operations of logarithms

How long does it take to grow money from the current amount of $1 to $4?

- ln(4) = 1.3862; so we have to wait 1.3862 unit of time.
- But the growth from $1 to $4 can happen in a complex situation such that the growth was doubling the amount at 2 time points, from $1 double to $2 and then from $2 double to $4. Thus, ln(4) = Time to double and double again;

  ln(4) = ln(2x2) = [ln(2)+ln(2)] = [0.6931+0.6931] = 1.3862

  Same answer as simple ln(4)!

How long does it take to grow money from the current amount of $1 to $6?

- ln(6) = 1.7917; so we have to wait 1.7917 unit of time.
- But, the growth from $1 to $6 can happen in a complex situation such that the growth was triple first and then double, from $1 triple to $3 and then from $3 double to $6. Thus, ln(6) = Time to triple and then double;

  ln(6)= ln(3 x 2) = [ln(3) + ln(2)] = [1.0986 + 0.6931] = 1.7917.

How long does it take to grow money from the current amount of $1 to $2?

- ln(2) = 0.6931
- But the growth from $1 to $2 can happen in a situation that the growth increase four times first

and then get decrease 2 times downwards, from $1 triple to $4 and from $4 doubling downwards to $2. Thus, ln(2) = Time to 4-times increase and then 2-times decrease;

ln(2) = ln (4/2) = [ln(4)-ln(2)] = [1.3862-0.6931] = 0.6931

- Or, the growth from $1 to $2 may occur in another different situation that the growth increase six times first and then get decrease 3 times downwards, from $1 increase 6-times to $6 and from $6 reduced down 3-times to $2. Thus, ln(2) = Time to 6-times increase and then 3-times decrease;

  ln(2) = ln(6/3) = [ln(6)-ln(3)] = [1.7917-1.0986] = 0.6931

The basic rules of logarithm turning "multiplication into addition" and "division into subtraction" are as follow:

ln(a x b) = ln(a)+ln(b)

ln (a/b) = ln(a)-ln(b)

In the next issue, we will discuss how statisticians use 'Log' in managing and analyzing data. (To be continued)

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